A Continuously Observed Two-level System Interacting with a Vacuum Field

R. Kullock[1] *and N. F. Svaiter* [2]

Centro Brasileiro de Pesquisas Físicas,

Rua Dr. Xavier Sigaud 150,

22290-180, Rio de Janeiro, RJ Brazil

Abstract

A discussion of the quantum Zeno effect and paradox is given. The quantum Zeno paradox claims that a continuously observed system, prepared in a state which is not an eigenstate of the Hamiltonian operator, never decays. To recover the classical behavior of unstable systems we consider a two-level system interacting with a Bose field, respectively prepared in the excited state and in the Poincaré invariant vacuum state. Using time-dependent perturbation theory, we evaluate for a finite time interval the probability of spontaneous decay of the two-level system. Using the standard argument to obtain the quantum Zeno paradox, we consider N measurements where $N \to \infty$ and we obtain that the non-decay probability law is a pure exponential, therefore recovering the classical behavior.

PACS numbers:03.70+k,04.62.+v

[1]e-mail: ricardokl@cbpf.br

[2]e-mail: nfuxsvai@cbpf.br

1 Introduction

In the present paper we are interested to show that a continuously observed quantum system has a classical time evolution behavior if it interacts with a vacuum field. Being more specific, we study the time evolution of a two-level system, i.e., a qubit, interacting with a Bose field prepared in the Poincaré invariant vacuum state, showing that the non-decay probability has a exponential behavior.

Unstable systems in quantum mechanics have been the subject of many investigations since the origin of this formalism, and have a long story starting with the seminal papers of Gamow [1] and Wigner [2]. The temporal evolution of these quantum systems can be roughly divided in three different behaviors. A gaussian-like behavior at short times, an exponential decay at intermediate times and finally a power-law decay at long times [3] [4] [5]. The gaussian behavior for short times is the key point of our discussions bellow, since it leads under general physical conditions to the inhibition of the decay of unstable quantum mechanical systems.

This result was obtained by Misra and Sudarshan [6] [7] using the von Neumann description of measurement processes [8]. These authors proved that the realization of many successive measurements dramatically slow down the evolution of an unstable system. In the limit of continuous observations, the temporal evolution of a quantum system can be frozen, defining respectively the quantum Zeno effect and paradox, after the paradox given by the Greek philosopher Zeno [9]. Actually, many years before, a close related result to the one obtained by Misra and Sudarshan was achieved by Khalfin [10], where a proof of the deviation from the exponential decay-law for large times was given.

The first conceptual question in the Misra and Sudarshan construction is the following: we may ask whether is possible to realize this limit of continuous observation. Although some authors argue that this limit of continuous observation is not physical and must be regarded as a mathematical idealization [11] [12] [13], we prefer to discuss this delicate issue later.

As has been stressed by many authors [14] [15] [16] [17] [18] [19] [20], any quantum mechanical system which we are interested to investigate its behavior on time interacts with the surroundings. In other words, in Nature we deal actually with open systems which are influenced by the surrounding world through exchange of energy, or in a more abstract way, information. These considerations motivate us to investigate unstable systems which are continuously observed in a finite time interval, including a vacuum field in the problem. The two-level system, prepared in a eigenstate of its free Hamiltonian, can make a transition to the lower eigenstate driven by the vacuum field, which acts as a reservoir. This is a development of an old idea. In the derivation of Planck's radiation law, Einstein introduced the idea of spontaneous emission, where a system makes a transition to a lower eigenstate without external stimulation [21].

In quantum mechanics, in treating arbitrary systems the time evolution of closed systems is described by one-parameter group of unitary operators, and the equation of motion of such systems are symmetric to time reversal. When a small system interacts with an reservoir, which is characterized by an infinitely large number of degrees of freedom, the time evolution of the small system can not be represented in terms of unitary Hamiltonian dynamics and we call it an open

quantum system. There are a variety of theoretical models of reservoirs. One situation is when the system S is coupled to an infinite number of harmonic oscillators. In this situation there are two kinds of reservoir of common interest. The first one is a thermal reservoir, where we assume that the harmonic oscillators are in thermal equilibrium at temperature β^{-1}. The second one is a squeezed reservoir. The specific system-reservoir model which is appropriate for the study of several interesting situations is when the harmonic oscillator bath is constituted by a Bose field in free space.

The aim of the paper is to recover the classical time evolution behavior for continuously observed systems. The situation that we are interested to study is when the two-level system is prepared in the excited state and it is interacting with a Bose field prepared in the Poincaré invariant vacuum state. Using perturbation theory we compute the probability of decay evaluated in a finite time interval.

These calculations are not new in the literature. Discussing model detectors, Svaiter and Svaiter [22] [23] assumed a weak-coupling between a two-level system and a Bose massless field. These authors evaluate the transition rates of the two-level system in different kinematic situations without use of the rotating-wave approximation [24] [25]. Further, Ford and collaborators [26], using the same model, assumed the presence of one or two infinite perfectly reflecting plates (mirrors). They show how these mirrors, which change the vacuum fluctuations associated to the Bose field, influence radiative processes at zero temperature. They also evaluate the probability per unit time of spontaneous emission at finite temperature. Radiative processes of atoms in waveguides and cavities also have been investigated by many authors. See for example the Ref. [27]. In the Refs. [28] [29] one of the authors continue to investigate radiative processes associated to the Unruh-DeWitt detector [30] [31] [32], in interaction with a massless scalar field. Being more precise, in Ref. [29], it was calculated the detector's excitation rate when it is uniformly rotating around some fixed point, when the scalar field is prepared in the Poicaré invariant vacuum state, and also when the detector is inertial and the field is prepared in the Trocheries-Takeno vacuum state [33] [34]. These two response functions allow to the authors to present questions analogous to those discussed by Mach in the Newton's bucket experiment in a quantum mechanical level.

This paper is organized as follows. In Section II we briefly discuss the theory of the classical and quantum mechanical decays and the quantum Zeno paradox. In Section III, assuming that the two-level system interact with a Bose field prepared in the Poincaré invariant vacuum state, the probability of decay of the two-level system is evaluated in a finite time interval. In Section IV, using the same arguments as in the quantum Zeno paradox, we obtain that in the case of continuous observations the non-decay probability law is exponential for all times. Conclusions are given in Section V. In the Appendix we discuss qubit-boson field interaction Hamiltonians. In the paper we use $k_B = c = \hbar = 1$

2 The classical and quantum mechanical decays and the quantum Zeno paradox

To introduce probability in classical physics we have to make use of a huge number of identical prepared systems. The classical theory of decay is quite simple and is based on the assumption that unstable systems have a certain probability of decay. The basic features of this simple model is that we assume a Markovian approximation. Therefore this probability does not depend on the past history of the unstable system. Let us assume N unstable systems, and that the decay probability per unit time be a constant that we call Γ. For simplicity Γ is characteristic of the system and also does not depend on the total number of unstable systems nor on the environment surrounding them. Let us define the number of unstable systems at time t by $N(t)$. Therefore the number of systems that will decay in the infinitesimal interval of time dt in $dN(t)$. Consequently we have

$$-dN(t) = N\,\Gamma\,dt. \tag{1}$$

Defining the inverse of Γ, i.e., the lifetime of the unstable system by τ_E ($\Gamma = \frac{1}{\tau_E}$), the number of unstable systems at a generic time t is

$$N(t) = N(0)\,\exp\left(-\frac{t}{\tau_E}\right), \tag{2}$$

where $N(0)$ is the number of unstable systems at the beginning of our observation, i.e., $t = 0$. One define the non-decay classical probability $P_{class}(t)$ as

$$P_{class}(t) = \frac{N(t)}{N(0)} = \exp\left(-\frac{t}{\tau_E}\right). \tag{3}$$

For short times ($t << \tau_E$) we can write

$$P_{class}(t) = 1 - \frac{t}{\tau_E} + ... \tag{4}$$

Note that we are excluding cooperative effects, therefore Γ and also $P(t)$ are environment-independent. The solution given by the Eq.(3) has a dissipative behavior and is a fundamental law that gives the classical behavior of unstable systems, as in experimental nuclear physics, for instance.

In quantum mechanics we introduce probability even working with a single system. A quantum mechanical treatment of the same problem give us a short and large time behaviors which are in disagreement with the exponential law obtained in Eq.(3). Let us first discuss the deviation from the exponential decay law for large times. We are following the arguments presented in the Ref. [7]. Let us assume a quantum system with a set of observables, i.e., operators which commute

with the Hamiltonian of the system. The Hamiltonian has a complete set of eigenstates, which is a basis of the Hilbert space, therefore every state vector of the system can be expressed in terms of it. For simplicity we assume that there is only one unstable state that we represent by $|\,a\,\rangle$, which is orthogonal to the bound states of the Hamiltonian operator H. Since the dynamic is time-translational invariant, the unitary operator $U(t_2 - t_1)$ propagates the system from t_1 to t_2. Using the self-adjoint operator H of the system, the dynamics is defined by the unitary operator $U(t) = e^{-iHt}$. Suppose that we are studying the temporal evolution of the system after $t = 0$. Let us define the spectral projection of the Hamiltonian operator by

$$H = \int d\lambda\, \lambda |\,\lambda\,\rangle\,\langle\,\lambda\,|. \tag{5}$$

The energy distribution function of the state $|\,a\,\rangle$ or the probability that the energy of the unstable state $|\,a\,\rangle$ lies in the interval $[E, E + dE]$ is given by

$$\int_{E}^{E+dE} \langle\, a|\,\lambda\rangle\,\langle\,\lambda|\,a\rangle\, d\lambda. \tag{6}$$

The non-decay probability at the time t is defined by $P_{quant}(t)$. Therefore, using the standard interpretation of quantum mechanics, we have that

$$P_{quant}(t) = |\langle\, a\,|\, e^{-iHt}\,|\,a\,\rangle|^2, \tag{7}$$

where the decay probability is given $[1 - P_{quant}(t)]$. Let us study the non-decay amplitude. It is given by

$$\langle\, a\,|\, e^{-iHt}\,|\,a\,\rangle = \int d\lambda\, e^{-i\lambda t}\,\langle\, a|\,\lambda\rangle\,\langle\,\lambda|\,a\rangle\, d\lambda. \tag{8}$$

If $\langle\,\lambda|\,a\rangle = 0$, for $\lambda < 0$, i.e., the spectrum of the Hamiltonian operator H is bounded from bellow, then when $t \to \infty$ the quantity $P_{quant}(t)$ decreases to zero less rapidly than any exponential of the form $e^{-\sigma t}$. Therefore we have a deviation from the exponential decay law at large times. Also, the treatment for the same problem for short time give us a short time behavior which is quadratic and therefore in disagreement with the exponential law obtained in Eq.(3).

Let us assume again a quantum system with an Hamiltonian operator H with a complete set of eigenstates denoted by $|\,i\,\rangle$ $(i = 1, 2, 3, ...)$. If we prepare the system in a normalized state $|\,a\,\rangle$ which is not an eigenstate of H, it is possible to show that the non-decay probability at short times is of the gaussian type. A short time expansion using the Eq.(7) yields

$$P_{quant}(t) = 1 - \frac{t^2}{\tau_z^2} + ..., \tag{9}$$

where the quantity $\tau_z^{-1} = [\langle\, a\,|\, H^2\,|\,a\,\rangle - \langle\, a\,|\, H\,|\,a\,\rangle^2]^{\frac{1}{2}}$ is the inverse of the characteristic time of the gaussian evolution. The crucial feature of this approximation is that the non-decay probability after a short observation time t is quadratic. The quantity τ_z is also called the Zeno time.

Let us assume that the quantum measurement occurs instantaneously. We also assume that is possible to perform infinitely many measurements in a given finite interval. Suppose that we perform N measurements at equal time interval which satisfies $T = N\Delta\tau$. In each measurement we observe that the system stays in the initial state $|\, a\,\rangle$ which was defined before. The probability of observing the initial state at the final time T after N measurements reads

$$P^{(N)}_{quant}(T) = \left[P_{quant}\left(\frac{T}{N}\right)\right]^N. \tag{10}$$

Substituting Eq.(9) in Eq.(10) we have

$$P^{(N)}_{quant}(T) \approx \left[1 - \frac{1}{\tau_z^2}\left(\frac{T}{N}\right)^2\right]^N. \tag{11}$$

For very large N we get

$$P^{(N)}_{quant}(T) \approx \, \exp\left(-\frac{T^2}{\tau_z^2 N}\right), \tag{12}$$

and repeated observations slow-down the evolution of the unstable system and increase the probability that the system remains in the initial state at T. If we are able to set $N \to \infty$ one obtains

$$\lim_{N\to\infty} P^{(N)}_{quant}(T) \approx \lim_{N\to\infty} \exp\left(-\frac{T^2}{\tau_z^2 N}\right) = 1. \tag{13}$$

This is a very simple derivation of the quantum Zeno paradox. The unstable quantum system becomes stable if we perform infinitely continuous measurements.

There are many physical assumptions that we have to make to obtain this effect. Many authors claim that the limit of infinite measurements is non-physical, and it is in contradiction with the Heiseberg uncertainty principle [35] [36]. We leave open these questions right now and we shall come back to this important issue when we discuss the interpretation of time-energy uncertainty relations [37] [38] [39] [40] [41].

In the next Section we are interested in the quantum measurement of a single object interactng with the vacuum modes and how to evaluate the probability of transition in a finite observation time. Since we are assuming a weak-coupling between the two-level system and the environment, the probability of decay of the two-level system is computed using first-order approximation in perturbation theory.

3 The probability of decay evaluated for a finite time interval

In this section we are interested in computing radiative processes of a quantum two-level system, interacting with a vacuum field. For simplicity we will use the following notation. The two energy levels of the two-level system, i.e., the ground and excited energy levels, are given by ω_g and ω_e ($\omega_e - \omega_g = \omega > 0$), with eigenstates of the free two-level system Hamiltonian $|\,g\,\rangle$ and $|\,e\,\rangle$, respectively. We are assuming a non-zero monopole matrix element between these two states and we can assume that the diagonal elements of the monopole operator vanish.

As we discuss in the appendix, the coupling between the massless scalar field and the two-level system is given by a monopole interaction Hamiltonian, i.e.,

$$H_I = \lambda\, m(\tau)\, \varphi(x(\tau)), \tag{14}$$

where $m(\tau)$ is the monopole operator of the two-level system, $\varphi(x(\tau))$ is the scalar field operator. The total hamiltonian of the system is given by Eq.(A.9), the free Hamiltonian of the scalar field and the Eq.(A.26), where λ is a small coupling constant between the qubit and the quantized Bose field.

We would like to stress that in general, measurement and state preparation are different phenomena. In quantum mechanics preparing a particular state might involve a special type of measurement, but there are an infinite number of prepared states which are not associated with measurements. For example, a generic state for the two-level system $|\,\Psi\rangle$ can be written as $|\,\Psi\rangle = \alpha\,|\,e\rangle + \beta|\,e\rangle$. The normalization condition gives $|\,\alpha|^2 + |\,\beta|^{\,2} = 1$. We can also prepare the two-level system in another state which is not an eigenstate of the Hamiltonian H_Q. Introducing the variables θ and ϕ we can write

$$|\,\Psi\rangle = |\,\theta,\phi\rangle = e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right)|\,e\rangle + e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right)|\,g\rangle. \tag{15}$$

Clearly $|\,\theta,\phi\rangle$ is a complete set, since $\int_0^\pi d\theta\sin\theta\int_0^{2\pi}\frac{d\phi}{2\pi}\,|\,\theta,\phi\rangle\,\langle\,\theta,\phi| = 1$. As already stated, we are interested to study the probability of decay from the excited state, driven by the vacuum fluctuations. Therefore we assume that the two-level system is in an eigenstate of the Hamiltonian H_Q. To evaluate the probability of decay (excitation) of the two-level system interacting with the Bose field, we can define the prepared initial state of the system in $\tau = 0$ as $|\,\tau_0\,\rangle = |\,e\,\rangle \otimes |\,\Phi_i\,\rangle$ ($|\,\tau_0\,\rangle = |\,g\,\rangle \otimes |\,\Phi_i\,\rangle$), where $|\,\Phi_i\,\rangle$ is the initial state of the field. Both situations can be analyzed using the same formalism. In the interacting picture, using the first-order approximation the probability of transition $P(E,\tau,0)$ after the time interval τ is given by

$$P(E,\tau,0) = \lambda^2|\langle\, e\,|\, m(0)\,|\,g\,\rangle|^{\,2}\, F(E,\tau,0), \tag{16}$$

where the response function $F(E,\tau,0)$ is given by

$$F(E,\tau,0) = \int_0^\tau d\tau' \int_0^\tau d\tau''\, e^{-iE(\tau'-\tau'')}\langle\,\Phi_i\,|\,\varphi(x(\tau'))\,\varphi(x(\tau''))|\,\Phi_i\,\rangle, \tag{17}$$

and in the above equation $E = \pm\,\omega$, where the signs $(+)$ and $(-)$ represent the excitation and decay process, respectively. Note that to obtain Eq.(17) we sum over all possible final states of the Bose field and we are using the completeness relation $\sum_f |\,\Phi_f\,\rangle\,\langle\,\Phi_f| = 1$, where $|\,\Phi_f\,\rangle$ is an arbitrary Bose field final state. We are obtaining information about the time evolution of the sub-system, i.e., the two-level system. This approach must be equivalent to study the density operator of the sub-system, i.e., the two-level system, which describes the dynamic evolution of this sub-system interacting with the environment through master equations. The equivalence between the density operator description and first-order perturbation theory to evaluate the quantum non-decay probability was demonstrated a long time ago by Fonda et al [3].

To proceed, let us suppose that it is possible to prepare the scalar field in Poincaré invariant vacuum state $|\,0, M\,\rangle$, or the Minkowski vacuum state. Therefore in the above equation the quantity $\langle\,\Phi_i\,|\,\varphi(x(\tau'))\,\varphi(x(\tau''))|\,\Phi_i\,\rangle$ becomes the positive Wightman function associated with the Bose scalar field evaluated in the world-line of the qubit. There are two points that we would like to stress. The first is that in the integrand of Eq.(17), the two-point correlation function depends only on the time difference $(\tau' - \tau'')$. The integration over τ' and τ'' is carried out over the square $0 \leq \tau' \leq \tau$, $0 \leq \tau' \leq \tau$ [22]. The second one is that we are not using the rotating-wave approximation, used by Glauber [42] and others, to define an ideal photo-counter detector. Therefore in the response function $F(E, \tau, 0)$ the vacuum fluctuations contributions associated with the Bose field are taken into account and we are studying the radiative processes associated to the qubit induced by a vacuum field.

To calculate the probability of transition evaluated in a finite time interval let us prepare the system in the initial instant of time τ_i in the state

$$|\,\tau_i\,\rangle = |\,e\,\rangle\,\otimes\,|\,0, M\,\rangle, \tag{18}$$

and assume that we observe the system in the ground state in the instant of time τ_f. In this situation the response function becomes

$$F^{(1)}(E, \tau_f, \tau_i) = \int_{\tau_i}^{\tau_f} d\tau' \int_{\tau_i}^{\tau_f} d\tau''\, e^{-iE(\tau' - \tau'')}\langle\,0, M\,|\,\varphi(x(\tau'))\,\varphi(x(\tau''))|\,0, M\,\rangle, \tag{19}$$

where $E = -\omega$. We are using the subscript (1) to call the attention that this quantity is evaluated in the first measurement. Defining $\Delta\tau = \tau_f - \tau_i$, and introducing the variables $\xi = \tau' - \tau''$ and $\eta = \tau' + \tau''$, the response function given by Eq.(19) can be written as:

$$F^{(1)}(E, \Delta\tau) = -\frac{1}{4\pi^2}\int_{-\Delta\tau}^{\Delta\tau} d\xi(\Delta\tau - |\,\xi|\,)\frac{e^{-iE\xi}}{(\xi - i\varepsilon)^2}, \tag{20}$$

where the $i\varepsilon$ is introduced to specify correctly the singularities of the Wightman function, to respect causality requirements. Let us split the response function $F^{(1)}(E, \Delta\tau)$ in two contributions:

$$F^{(1)}(E, \Delta\tau) = F_1^{(1)}(E, \Delta\tau) + F_2^{(1)}(E, \Delta\tau), \tag{21}$$

where the functions $F_1^{(1)}(E,\Delta\tau)$ and $F_2^{(1)}(E,\Delta\tau)$ are given respectively by

$$F_1^{(1)}(E,\Delta\tau) = -\frac{1}{4\pi^2}\int_{-\Delta\tau}^{\Delta\tau} d\xi\Delta\tau\frac{e^{-iE\xi}}{(\xi - i\varepsilon)^2} \tag{22}$$

and

$$F_2^{(1)}(E,\Delta\tau) = \frac{1}{4\pi^2}\int_{-\Delta\tau}^{\Delta\tau} d\xi\,|\,\xi|\frac{e^{-iE\xi}}{(\xi - i\varepsilon)^2}. \tag{23}$$

After some calculations [22] [23] we obtain that the functions $F_1^{(1)}(E,\Delta\tau)$ and $F_2^{(1)}(E,\Delta\tau)$ can be written as:

$$F_1^{(1)}(E,\Delta\tau) = \frac{\Delta\tau}{2\pi}\left(-E\,\Theta(-E) + \frac{\cos E\Delta\tau}{\pi\Delta\tau} + \frac{|E|}{\pi}\left(Si|E|\Delta\tau - \frac{\pi}{2}\right)\right) \tag{24}$$

and

$$F_2^{(1)}(E,\Delta\tau) = \frac{1}{2\pi^2}\left(-\gamma + Ci|E|\Delta\tau - \ln\,\varepsilon|E| - 1\right). \tag{25}$$

In the Eq.(24) and Eq.(25), γ is the Euler constant and the $Si(z)$ and $Ci(z)$ functions are defined respectively by [43]:

$$Si(z) = \int_0^z \frac{\sin\,t}{t}\,dt, \tag{26}$$

and

$$Ci(z) = \gamma + \ln z + \int_0^z \frac{1}{t}\left(\cos t - 1\right)dt. \tag{27}$$

The Eq.(25) has two divergences. One given by $\ln\Delta\tau$ as $\Delta\tau \to 0^+$ and other given by $\ln\varepsilon$. In a full perturbative renormalizable quantum field theory, there is a regularization and also a renormalization procedure, where the infinities can be eliminated. One way to circumvent this problem is to define the rate $R^{(1)}(E,\Delta\tau) = \frac{d}{d(\Delta\tau)}F^{(1)}(E,\Delta\tau)$ [44]. Since we are interested in the non-decay probability in a finite time, let us define the renormalized probability of transition $P_{ren}^{(1)}(E,\Delta\tau)$ by

$$P_{ren}^{(1)}(E,\Delta\tau) = \lambda^2|\langle\, e\,|\,m(\tau_i)\,|\,g\,\rangle|^{\,2}\left(F^{(1)}(E,\Delta\tau) - \ln\frac{\Delta\tau}{\varepsilon}\right). \tag{28}$$

To support this procedure we can use the argument that these divergences are spurious and can not appear in the physically measured processes. Using this renormalization procedure, the probability of decay can be written as

$$P_{ren}^{(1)}(E,\Delta\tau) = \lambda^2|\langle\, e\,|\,m(\tau_i)\,|\,g\,\rangle|^{\,2}\,F_{ren}^{(1)}(E,\Delta\tau), \tag{29}$$

where $F^{(1)}_{ren}(E,\Delta\tau)$ is given by

$$F^{(1)}_{ren}(E,\Delta\tau)=\frac{1}{2\pi^2}\left(|E|\Delta\tau\left(\frac{\pi}{2}+Si\,|E|\,\Delta\tau\right)+\cos\,E\Delta\tau-1+\int_0^{|E|\,\Delta\tau}\frac{1}{\xi}\left(\cos\xi-1\right)d\xi\right). \quad (30)$$

For a small time interval $\Delta\tau$ the transition probability contains two contributions: the first one that increases linearly with the time interval and the second one that increases quadratically with the time interval. The probability of non-decay of this two-level system after a finite time interval $\Delta\tau$ is given by $P^{(1)}_{still}(E,\Delta\tau)=(1-P^{(1)}_{ren}(E,\Delta\tau))$.

Instead of using the interaction picture and the perturbation theory in first-order approximations, it is possible to use the Heisenberg equations of motion to the Dicke operators and also to the annihilation and creation operators associated to the Bose field [45] [46]. Clearly both methods of calculations must give identical results.

In the next Section we use this probability of non-decay to show that the observed probability is exponential of the time T. There are some technical problems in the second measurement. Although we assume that the interaction between the qubit and the field is weak, we can not suppose that the state of the field does not change in time. We conclude that to study the time evolution of the system in the second measuremet, we have to assume that the qubit is still in the excited state $|\,e\,\rangle$, and the Bose field is in an arbitrary state $|\,\Phi_i\rangle$. We will assume for simplicity that the initial sate of the field in the second measurement is a many particle state.

4 The exponential decay after N successive measurements

The aim of this section is to show that if we couple the two-level system with the Bose field in the vacuum state we recover the exponential non-decay probability if the system is continuously observed.

Suppose that we perform N measurements at equal time interval which satisfies $T=N\Delta\tau$, and in each measurement the system stays in the initial excited state $|\,e\,\rangle$ defined before. Again, we would like to stress that to obtain the Eq.(17) we assumed that no observation was made to discriminate among possible final Bose field states. Therefore to calculate the probability of observing the initial state of the qubit $|\,e\,\rangle$, at the final time T after N measurements we have to point out the following fact. In the first measurement, to obtain the probability $P^{(1)}_{still}(E,\Delta\tau)=(1-P^{(1)}_{ren}(E,\Delta\tau))$, where $P^{(1)}_{ren}(E,\Delta\tau)$ is given by Eq.(29) and Eq.(30), we summed over all possible final states of the Bose field. This reflect the fact that we are interested in the final state of the qubit and not that of the field. For this reason, in the second measurement we can not assume that the initial state of the Bose field is identical with the initial state of the field prepared in the first time interval $\Delta\tau$.

Before the second measurement, the information of the state of the field is retained by the system. Since the initial Bose field state in the second measurement is indeterminate for us, we choose an arbitrary state $|\,\Phi_i\,\rangle$. Therefore to study the time evolution of the system in the second measuremet, let us suppose that the qubit is still in the excited state $|\,e\,\rangle$, and the Bose field is in an arbitrary state $|\,\Phi_i\,\rangle$. To find the probability of transition from the state $|\,e\,\rangle \otimes |\,\Phi_i\,\rangle$ to the final state $|\,g\,\rangle \otimes |\,\Phi_f\,\rangle$ after the second time interval $\Delta\tau$ we have to evaluate the expression $P^{(2)}(E, \tau_f + \Delta\tau, \tau_i + \Delta\tau)$ given by

$$P^{(2)}(E, \tau_f + \Delta\tau, \tau_i + \Delta\tau) = \lambda^2 |\langle\, e\,|\, m(\tau_i + \Delta\tau)\,|\, g\,\rangle|^{\,2}\, F^{(2)}(E, \tau_f + \Delta\tau, \tau_i + \Delta\tau), \tag{31}$$

where the new response function $F^{(2)}(E, \tau_f + \Delta\tau, \tau_i + \Delta\tau) = F^{(2)}(E, \Delta\tau)$ can be written as

$$F^{(2)}(E, \Delta\tau) = \int_{-\Delta\tau}^{\Delta\tau} d\xi (\Delta\tau - |\,\xi|\,) e^{-iE\xi} \langle\, \Phi_i\,|\, \varphi(x(\tau'))\, \varphi(x(\tau''))|\, \Phi_i\,\rangle. \tag{32}$$

Note that we are using the same convention used in the section IV. We have again that $\Delta\tau = \tau_f - \tau_i$, and $\xi = \tau' - \tau''$. Note that to obtain the Eq.(32) we are using again the completeness relation over the final states of the field in the second measurement.

We have a considerable arbitrariness in the choice of the initial state of the field after the first measurement. An appropriate starting point is to suppose that the initial state of the field in the second measurement is a many particle state with n_1 quanta with momenta $\mathbf{k_1}$ and energy ω_1, n_2 quanta with momenta $\mathbf{k_2}$ and energy ω_2 and so on. Therefore the two-point correlation function that appears in the Eq.(32) can be written as

$$\langle\, \Phi_i\,|\, \varphi(x(\tau'))\, \varphi(x(\tau''))|\, \Phi_i\,\rangle = \langle n_1(\mathbf{k}_1)...n_j(\mathbf{k}_j)|\, \varphi(x(\tau'))\, \varphi(x(\tau''))|\, n_1(\mathbf{k}_1)...n_j(\mathbf{k}_j)\rangle. \tag{33}$$

Using Eq.(33) it's not difficult to show that we can write the two-point correlation function $\langle\, \Phi_i\,|\, \varphi(x(\tau'))\, \varphi(x(\tau''))|\, \Phi_i\,\rangle$ in the following way:

$$G^+(x(\tau'), x(\tau'')) + \sum_i n_i u_{\mathbf{k}_i}(x(\tau')) u^*_{\mathbf{k}_i}(x(\tau'')) + \sum_i n_i u^*_{\mathbf{k}_i}(x(\tau')) u_{\mathbf{k}_i}(x(\tau'')), \tag{34}$$

where $G^+(x(\tau'), x(\tau''))$ is the positive Wightman function evaluated in the world line of the qubit and n_i is the number density of quanta in the k-space. The set $(u^*_{\mathbf{k}_i}(x), u_{\mathbf{k}_i}(x))$ is a basis in the space of solutions of the Klein-Gordon equation. Without loss of generality we can choose the plane-waves for the basis.

Taking the continuous limit, assuming that the quanta are distributed isotropically, and substituting Eq.(33) and Eq.(34) in Eq.(32) we have that the response function $F^{(2)}(E, \Delta\tau)$ in the second measurement can be written as

$$F^{(2)}(E, \Delta\tau) = F_1^{(2)}(E, \Delta\tau) + F_2^{(2)}(E, \Delta\tau). \tag{35}$$

In the Eq.(35), the quantity $F_1^{(2)}(E,\Delta\tau)$ is the vacuum contribution given by Eq.(20) and the quantity $F_2^{(2)}(E,\Delta\tau)$ is the non-vacuum contribution given by

$$F_2^{(2)}(E,\Delta\tau) = \frac{1}{4\pi^2}\int_{-\Delta\tau}^{\Delta\tau} d\xi\,(\Delta\tau - |\,\xi|\,)e^{-iE\xi}\,g(\xi). \tag{36}$$

In the above expression the function $g(\xi)$ that depends on the the number density of quanta in the k-space in the continuous limit is

$$g(\xi) = \int d\omega\,\omega\,n(\omega)\left(e^{i\omega\xi} + e^{-i\omega\xi}\right). \tag{37}$$

To proceed, we can extend the integration over all frequencies in the Eq.(37) and also replace the number density of quanta in the k-space, $n(\omega)$ by a constant value in the interval $[0,a]$. Using that [47]

$$\int_{-\infty}^{\infty} dx\, f_n(x)\, e^{ix\xi} = \frac{n!}{(-i\xi)^{n+1}}\left(1 + e^{ia\xi}\sum_{k=0}^{n}\frac{(-ia\xi)^k}{k!}\right), \tag{38}$$

where $f_n(x) = x^n$ for $0 < x < a$ and zero otherwise and $n = 1, 2, ...$ we get that Eq.(37) can be written as

$$g(\xi, a) = -\frac{2}{\xi^2} - \frac{1}{\xi^2}\left(e^{ia\xi} + e^{-ia\xi}\right) - \frac{2a}{\xi}\,\sin\xi a. \tag{39}$$

The first term in the above equation will give a contribution to the response function which is proportional to the one of the vacuum field. To proceed, let us substitute Eq.(39) in Eq.(36). Therefore we have that $F_2^{(2)}(E,\Delta\tau)$ can be written as

$$F_2^{(2)}(E,a,\Delta\tau) = f_1^{(2)}(E,\Delta\tau) + f_2^{(2)}(E,a,\Delta\tau) + f_3^{(2)}(E,a,\Delta\tau) + f_4^{(2)}(E,a,\Delta\tau)), \tag{40}$$

where $f_1^{(2)}(E,\Delta\tau) = 2F_1^{(2)}(E,\Delta\tau)$, and the two other terms in the above equation are given respectively by

$$f_2^{(2)}(E,a,\Delta\tau) = -\frac{1}{4\pi^2}\int_{-\Delta\tau}^{\Delta\tau} d\xi\,(\Delta\tau - |\,\xi\,|)\frac{e^{-i(E+a)\xi}}{(\xi - i\epsilon)^2}, \tag{41}$$

and $f_3^{(2)}(E,a,\Delta\tau) = f_2^{(2)}(E,-a,\Delta\tau)$. Again the $i\varepsilon$ is introduced to respect causality requirements. Finally, the function $f_4^{(2)}(E,a,\Delta\tau)$ is given by

$$f_4^{(2)}(E,a,\Delta\tau) = -\frac{a}{4\pi^2}\int_{-\Delta\tau}^{\Delta\tau} d\xi(\Delta\tau - |\,\xi\,|)\,e^{-iE\xi}\,\frac{\sin\xi a}{(\xi - i\epsilon)}. \tag{42}$$

The integral above can be written in the following form:

$$f_4^{(2)}(E,a,\Delta\tau) = -\frac{a\Delta\tau}{8\pi i}\left(\int_{-\Delta\tau}^{\Delta\tau}\frac{d\xi}{\xi}\,e^{-i\xi(E-a)} - \int_{-\Delta\tau}^{\Delta\tau}\frac{d\xi}{\xi}\,e^{-i\xi(E+a)}\right) + \frac{a}{2\pi}\int_0^{\Delta\tau} d\xi\sin\xi a\cos\xi E. \tag{43}$$

To carry out the integrations we use the fact that the two integral in the right side of the Eq.(43) can be interpreted as the principal value, i.e.

$$\epsilon(x) = \frac{1}{i\pi} P \int_{-\infty}^{\infty\tau} \frac{d\xi}{\xi}\, e^{i\xi x} \tag{44}$$

and also the last integral of the Eq.(43) can be carry out immediately and gives

$$\int_0^{\Delta\tau} d\xi \sin\xi a \cos\xi E = \frac{1}{2(E-a)}\Big(\cos(E-a)\Delta\tau - 1\Big) - \frac{1}{2(E+a)}\Big(\cos(E+a)\Delta\tau - 1\Big). \tag{45}$$

We conclude that in this second small time interval $\Delta\tau$ the transition probability contains also two contributions: the first one that increases linearly with the time interval and the second one that increases quadratically with the time interval. Following this line, the probability of observing the initial state at the final time T after N measurements reads

$$P_{still}^{(N)}(E,T) = \left[P_{still}^{(1)}\left(E, \frac{T}{N}\right)\right]\left[P_{still}^{(2)}(E,a,\frac{T}{N})\right]^{(N-1)}. \tag{46}$$

Using the fact that $P_{still}^{(i)}\left(E, \left(\frac{T}{N}\right)\right) = \left[1 - P_{ren}^{(i)}\left(E, \left(\frac{T}{N}\right)\right)\right]$ for $i = 1, 2$, we have

$$P_{still}^{(N)}(E,T) = \left[1 - P_{ren}^{(1)}\left(E, \left(\frac{T}{N}\right)\right)\right]\left[1 - P_{ren}^{(2)}(E,a,\frac{T}{N})\right]^{(N-1)}. \tag{47}$$

For $\Delta\tau = \frac{T}{N}$, expanding for small arguments and keeping terms only through order $\left(\frac{T}{N}\right)^2$ the quantity $F_{ren}^{(1)}(E,\Delta\tau)$ is written as

$$F_{ren}^{(1)}\left(E, \frac{T}{N}\right) \approx \left(\frac{|E|T}{4\pi N} + \frac{\alpha E^2 T^2}{4\pi N^2}\right), \tag{48}$$

where $\alpha = (\frac{1}{2} - \frac{3}{2\pi})$. The quantity $F_{ren}^{(2)}(E,\Delta\tau)$ also in the same order of the approximation can be written as

$$F_{ren}^{(2)}\left(E,a, \frac{T}{N}\right) \approx \left(p\,(E,a)\frac{T}{N} + q(E,a)\frac{T^2}{N^2}\right), \tag{49}$$

where $p\,(E,a)$ and $q\,(E,a)$ are functions of E and a given by

$$p\,(E,a) = \frac{1}{4\pi}\Big(3|E| + |E+a| + |E-a| + a\,\theta(a-E) - a\,\theta(-E-a) + a\,\theta(E+a)\Big) \tag{50}$$

and

$$q\,(E,a) = \frac{1}{8\pi^2}\left(5E^2(\pi-3) - 2a^2\right). \tag{51}$$

It is easy to see that, if $|E| < |\,a|$, then

$$p\,(E,a) = \frac{1}{4\pi}\,(3|E| + 4a)\,. \tag{52}$$

For $|E| > |\,a|$:

$$p\,(E,a) = \frac{5|E|}{4\pi}. \tag{53}$$

Finally for $|E| = |\,a|$, we obtain

$$p\,(E,a) = \frac{|E|}{\pi} \tag{54}$$

and

$$q\,(E,a) = (5\pi - 17)\frac{E^2}{8\pi^2}. \tag{55}$$

Defining $\lambda^2|\langle\, e\,|\, m(\tau_i)\,|\, g\,\rangle|^{\,2} = \sigma$, we may write the probability of observation the initial state at a finite time T after N measurements $P^{(N)}_{still}(E, \frac{T}{N})$ as

$$P^{(N)}_{still}(E,a,N) \approx \left[1 - \frac{1}{N}\left(p\,(E,a,\sigma)T + q(E,a,\sigma)\frac{T^2}{N}\right)\right]^N. \tag{56}$$

At this moment we would like to discuss the interpretation of time-uncertainty relations. There are a large amount of literature devoted to the interpretation of quantum mechanics. Nevertheless, concerning the time-uncertainty relations, there are a few papers discussing the implications of such relations. Landau and Peierls [37] and also Landau and Lifshitz [38] claim that the energy of a quantum system can be measured exactly at a given time. Nevertheless, we must take into account the change caused by the process of measurement. In the relation $\Delta E\,\Delta\tau > 1$, the quantity ΔE is the difference between two exactly measured energy values at two different instants of time, where $\Delta\tau$ is the time interval between the measurements. If we accept this interpretation, there is a finite but very large N constrained by an upper bound given by the Landau, Peierls and Lifshitz interpretation of the time-energy uncertainty relation ($N < TE$), and we obtain that the non-decay probability is polinomial but very similar to the exponential behavior.

On the other hand, this interpretation of the time-energy uncertainty relation is not universally accepted. Aharonov and Bohm [40] [41] claim that the time-uncertainty relations are not consistent with the general principles of quantum mechanics which require that the uncertainty relations be expressible in terms of operators. Therefore, they concluded that the energy of a quantum system can be measured in an arbitrary short time. In this framework we are able to take the limit $N \to \infty$, and we get

$$P^{(N)}_{still}(E,a,T) = \exp\left(-p\,(E,a,\sigma)T - q(E,a,\sigma)\frac{T^2}{N}\right). \tag{57}$$

To conclude, we obtain that repeated observations slow-down the evolution of the unstable system and increase the probability that the system remains in the initial state at T. If we are able to perform only a finite but very large N number of measurements in a finite time, constrained by an upper bound given by the Landau, Peierls and Lifshitz interpretation of the time-energy uncertainty relation, we obtain that the non-decay probability is polinomial but very similar to the exponential behavior. On the other hand, if we are able to perform continuous observation and the limit $N \to \infty$ can be used, the non-decay probability becomes a pure exponential.

To obtain this classical behavior, we follow different steps. First, we prepare the composed system in an initial state $|\,e\,\rangle \otimes |\,0, M\,\rangle$, i.e., we couple the two-level system with a vacuum field. Second, that the system evolves under the influence of the unobserved Bose field. Finally that we are able to perform continuous observations in the two-level system.

5 Conclusions

In this paper we study the time evolution of unstable systems after repeated but finite observations and also in the limit of continuous observation. We show that a continuously observed quantum system has a classical time evolution behavior if it interacts with a unobserved vacuum field.

Using perturbation theory in first-order approximation where a two-level system is interacting with the Bose field in the Poincaré invariant vacuum state we obtain two distinct types of behaviors. First a finite but very large N constrained by an upper bound given by the Landau, Peierls and Lifshitz interpretation of the time-energy uncertainty relation, and second, the case where $N \to \infty$, allowed by the Aharanov-Bohm interpretation of the same relation. Studying the non-decay probability in both situations, we obtain that the non-decay probability is polynomial and very similar to the exponential behavior for the first case. For the second case the non-decay probability is given by

$$P^{(N)}_{still}(E,T) = \exp\left(-\frac{T}{\tau_c}\right), \tag{58}$$

where $\tau_c \propto \frac{1}{E}$. It is important to remind that in the interaction Hamiltonian for our model, we are not assuming the rotating-wave-approximation, that excludes terms that represent simultaneous qubit and field excitation and de-excitation respectively. Although these terms are not-energy conserving, representing virtual processes, we understand that only for very large time interval $\Delta\tau$ the contribution coming from these terms can be neglected. In the case that we are interested, i.e., a small $\Delta\tau$, a more carefully procedure is not to omit the energy-non-conserving terms. This procedure allow us to obtain the above equation that accounts very well for experimental observed facts, as for example the decay of many quantum systems, as unstable atoms or nuclei. The result establishes that the quantum theory allow us to recover classical behavior under suitable circumstances.

From the preceding sections it is seen that, although our model of the qubit-Bose field composed system is quite satisfactory to obtain the experimentally observed non-decay probability law with

the exponential behavior, the approach of the paper is intrinsically limited since we would like to predict the same behavior in quite general quantum systems. We know that many systems have a complete set of discrete eigenstates but also a continuum spectrum. Therefore to construct a more realistic model to study radiative processes of unstable systems we have to generalize our model to one with two bound states and also a continuum of states. For example in the case of the atom, which is a practical photo-detector, there is a continuum of final electron states. Assuming the same two-levels and a continuum of states $|\,\omega_a\rangle$, with energy in the range $[\,\omega_c, \infty\,]$, and preparing the small system in the state$|\,e\rangle$ and again the Bose field in the vacuum state we have that the probability, evaluated for a finite time interval, of the system to makes a transition to the continuum is given by

$$P(\Delta\tau) = \int_{\omega_c}^{\infty} d\omega_a\, \rho(\omega_a) P(\omega_{ae}, \Delta\tau), \tag{59}$$

where $\omega_{ae} = \omega_a - \omega_e$ and $\rho(\omega_a)$ is a density of final excited states. Again, the quantity $P(\omega_{ae}, \Delta\tau)$ is given by

$$P_{ren}(\omega_{ae}, \Delta\tau) = \lambda^2 |\langle\, a\,|\, m(\tau_i)\,|\, e\,\rangle|^{\,2}\, F_{ren}(\omega_{ae}, \Delta\tau). \tag{60}$$

Note that we have to choose a particular form to the density of final excited states $\rho(\omega_{ae})$, to make sure that the integral given by Eq.(59) converges at infinity. This generalization is under investigation by the authors.

There are also different directions for investigation. To mention a few: first is to assume a strong-coupling between the qubit and the Bose field [48] [49]. Second, still in the weak-coupling regime, is to assume that the reservoir is in thermal equilibrium or in a squeezed state [50]. It can be shown that the behavior of the two-level system in a squeezed bath depends on the way in which the squeezed bath is prepared, showing Zeno or anti-Zeno effects. Also it is interesting to consider N qubits ($N \to \infty$) interacting with one mode of the Bose field, analyzing the situation where the qubit system acts as a reservoir whereas the Bose field is an open system, and study the dynamics of the reduced system.

Finally, this two-level system, referred to as a qubit is the elementary building block of a quantum computer [51] [52] [53] [54] [55]. This new area of research has revived the interest in open quantum systems. The fundamental technological problem is if it is possible to create entanglement properties of states in systems that interact with a reservoir. Several situations of entangled systems have been proposed, as for example involving trapping and also cooling a small number of atoms. How to isolate atoms from the environment in order to make the effect of decoherence negligible is an open problem until now.

Therefore, another natural extension of this paper is to generalize some results of the paper in the case of two-atom systems prepared in an entangled state [56]. Using time-dependent perturbation theory in a first-order approximation, evaluate the probability per unit-time of decay of the symmetric and anti-symmetric states given by Eq.(A.17) and Eq.(A.18) respectively to the ground state $|\,g_1\,\rangle \otimes |\,g_2\,\rangle$. Note that we have to continue to assume that the qubits also interact with the vacuum modes. The possibility to prepare the two-atom system in a entangled decoherence-free

state is a question that has fundamental importance in quantum computing applications. Although the spontaneous emission from a pair of atoms [57] [58] and the causal aspects of their spontaneous decay [59] [60] have been studied by some authors, the extension of this formalism for two-atom prepared in an entangled state, evaluating the probability of decay in a finite time interval is new in the literature.

6 Acknowlegements

We would like to thank S. Joffily, H. J. Mosqueira Cuesta and G. Flores-Hidalgo for enlightening discussions. This paper was supported by Conselho Nacional de Desenvolvimento Cientifico e Tecnológico do Brazil (CNPq).

A The qubit-Bose field interaction Hamiltonians

In this appendix we consider a very general situation where the system under investigation contains a large number of non-identical two-level systems. In order to describe the dynamics of the reservoir and the two-level systems we have to introduce the Hamiltonian governing the interaction of the quantized Bose field with free qubits. Free means that there is no interaction between the qubits. Therefore let us consider a Bose quantum system B, with Hilbert space $\mathcal{H}^{(B)}$ which is coupled with N qubits, with Hilbert space $\mathcal{H}^{(Q)}$. Let us assume that the reservoir is in thermal equilibrium at temperature β^{-1}. The Bose quantum system is a sub-system of the total system living in the tensor product space $\mathcal{H}^{(B)} \otimes \mathcal{H}^{(Q)}$.

Let us denote by H_B the Hamiltonian of the quantized Bose field, by H_Q the free Hamiltonian of the N-qubits and H_I the Hamiltonian describing the interaction between the quantized Bose field and the N qubits. The Hamiltonian for the total system can be written as

$$H = H_B \otimes I_Q + I_B \otimes H_Q + H_I, \tag{A.1}$$

where I_B and I_Q denotes the identities in the Hilbert spaces of the quantized Bose field and the N qubits.

The main purpose of this appendix is to discuss qubit-Bose field interaction Hamiltonians. Therefore, let us introduce the Dicke operators to describe each qubit. The free $j-th$ qubit Hamiltonian will be denoted by $H_D^{(j)}$, since we are using the Dicke representation. Therefore, we have

$$H_D^{(j)}|\,i\,\rangle_j = \omega_i^{(j)}|\,i\,\rangle_j, \tag{A.2}$$

where $|\,i\,\rangle_j$ are orthogonal energy eigenstates accessible to the $j-th$ qubit and $\omega_i^{(j)}$ are the respective eigenfrequencies. Using Eq.(A.2) and the orthonormality of the energy eigenstates we

can write the $j-th$ qubit Hamiltonian $H_D^{(j)}$ as

$$H_D^{(j)} = \sum_{i=1}^{2} \omega_i^{(j)} (|\, i\,\rangle \, \langle\, i|\,)_j. \tag{A.3}$$

Let us define the Dicke operators $\sigma^z_{(j)}$, $\sigma^+_{(j)}$ and $\sigma^-_{(j)}$ for each qubit by

$$\sigma^z_{(j)} = \frac{1}{2}\,(|2\,\rangle\,\langle\, 2| - \,|1\,\rangle\,\langle\, 1|\,)_j, \tag{A.4}$$

$$\sigma^+_{(j)} = \,(|2\,\rangle\,\langle\, 1|\,)_j, \tag{A.5}$$

and finally

$$\sigma^-_{(j)} = \,(|1\,\rangle\,\langle\, 2|\,)_j. \tag{A.6}$$

The Dicke representation is a second quantization of the qubits. Combining Eq.(A.3) and Eq.(A.4), the $j-th$ qubit Hamiltonian can be written as

$$H_D^{(j)} = \,\Omega^{(j)}\,\sigma^z_{(j)} + \frac{1}{2}\Big(\omega_1^{(j)} + \omega_2^{(j)}\Big), \tag{A.7}$$

where the energy gap between the energy eigenstates of the $j-th$ qubit is given by

$$\Omega^{(j)} = \omega_2^{(j)} - \omega_1^{(j)}. \tag{A.8}$$

Shifting the zero of energy to $\frac{1}{2}(\omega_1^{(j)} + \omega_2^{(j)})$ for each qubit, the $j-th$ qubit Hamiltonian given by Eq.(A.7) can be rewritten as

$$H_D^{(j)} = \Omega^{(j)}\,\sigma^z_{(j)}. \tag{A.9}$$

Note that the operators $\sigma^+_{(j)}$, $\sigma^-_{(j)}$ and $\sigma^z_{(j)}$ satisfy the standard angular momentum commutation relations corresponding to spin $\frac{1}{2}$ operators, i.e.,

$$\Big[\sigma^+_{(j)}, \sigma^-_{(j)}\Big] = 2\,\sigma^z_{(j)}, \tag{A.10}$$

$$\Big[\sigma^z_{(j)}, \sigma^+_{(j)}\Big] = \sigma^+_{(j)}, \tag{A.11}$$

and finally

$$\Big[\sigma^z_{(j)}, \sigma^-_{(j)}\Big] = -\sigma^-_{(j)}. \tag{A.12}$$

A well-known model is a combining system where we have only one mode of the quantized field. The Hamiltonian of the $j-th$ qubit $H_D^{(j)}$, with the contribution of the one-mode quantized Bose

field H_S, and the interaction Hamiltonian $H_I^{(j)}$, can be used to define the Hamiltonian of the total system, given by

$$I_B \otimes H_D^{(j)} + H_B \otimes I_Q + H_I^{(j)} =$$
$$I_B \otimes \Omega^{(j)}\, \sigma^z_{(j)} + \omega_0\, a^\dagger a \otimes I_Q + g\left(a + a^\dagger\right) \otimes \left(\sigma^+_{(j)} + \sigma^-_{(j)}\right), \tag{A.13}$$

where the second term in the Eq.(A.13) has the contribution from the quantized Bose field single mode Hamiltonian and the last term is the interaction Hamiltonian of the $j-th$ qubit with the one-mode quantized field. In the Eq.(A.13) g is a small coupling constant between the qubit and the one mode quantized Bose field. The generalization to N qubits is described by

$$I_B \otimes \sum_{j=1}^{N} H_D^{(j)} + H_B \otimes I_Q + \sum_{j=1}^{N} H_I^{(j)} =$$
$$I_B \otimes \sum_{j=1}^{N} \Omega^{(j)}\, \sigma^z_{(j)} + \omega_0\, a^\dagger a \otimes I_Q + \left(a + a^\dagger\right) \otimes \frac{g}{\sqrt{N}} \sum_{j=1}^{N} \left(\sigma^+_{(j)} + \sigma^-_{(j)}\right), \tag{A.14}$$

where the first summation in the right hand side

$$\sum_{j=1}^{N} \Omega^{(j)}\, \sigma^z_{(j)} = \Omega^{(1)}\, \sigma^z_{(1)} \otimes \mathbf{1} \otimes ... \otimes \mathbf{1} + ... + \mathbf{1} \otimes \mathbf{1} \otimes ... \otimes \mathbf{1} \otimes \Omega^{(N)}\, \sigma^z_{(N)}, \tag{A.15}$$

and $\mathbf{1}$ denotes the identity in the Hilbert space of each qubit. We can also introduce a qubit-qubit interaction, which is relevant in the study of entangled states. In an entangled system, the state of the composite system can not be factorized in to a product of the states of its sub-systems. For example in the case of two-atom systems, takes the form

$$H_{(qq)} = \sum_{i \neq j}^{2} H_{(ij)}\, \sigma^+_{(i)} \otimes \sigma^-_{(j)}. \tag{A.16}$$

In the absence of the "dipole-dipole" interaction the pure Hilbert space of the two qubit system is spanned by the states $|\, g_1 \rangle \otimes |\, g_2 \rangle, |\, g_1 \rangle \otimes |\, e_2 \rangle, |\, e_1 \rangle \otimes |\, g_2 \rangle, |\, e_1 \rangle \otimes |\, e_2 \rangle$, where g and e denotes respectively the ground and the excited state. If we include the "dipole-dipole" interaction term in the form of Eq.(A.16), the vectors $|\, g_1 \rangle \otimes |\, e_2 \rangle$ and $|\, e_1 \rangle \otimes |\, g_2 \rangle$ are not more eigenstates of the Hamiltonian of the qubits systems. It can be shown that these two vectors states must be substituted by the two entangled states, known in the literature as maximally entangled states [56]

$$|\, s \rangle = \frac{1}{\sqrt{2}} \left(|\, e_1 \rangle \otimes |\, g_2 \rangle + |\, g_1 \rangle \otimes |\, e_2 \rangle\right) \tag{A.17}$$

and

$$| \, a \, \rangle = \frac{1}{\sqrt{2}} \left(| \, e_1 \, \rangle \, \otimes \, | \, g_2 \, \rangle - | \, g_1 \, \rangle \, \otimes \, | \, e_2 \, \rangle \right). \tag{A.18}$$

Going back to Eq.(A.14), the interaction Hamiltonian is simplified if we assume the Jaynes-Cummings model [61]. Considering the Jaynes-Cummings model for one qubit, we have

$$\begin{aligned} & I_B \, \otimes \, H_D^{(j)} + H_B \, \otimes \, I_Q + H_I^{(j)} = \\ & I_B \, \otimes \, \Omega^{(j)} \, \sigma^z_{(j)} + \omega_0 \, a^\dagger a \, \otimes \, I_Q + g\Big(a \, \otimes \, \sigma^+_{(j)} + a^\dagger \, \otimes \sigma^-_{(j)}\Big). \end{aligned} \tag{A.19}$$

The generalization to N qubits is straightforward and is given by

$$\begin{aligned} & I_B \, \otimes \sum_{j=1}^{N} H_D^{(j)} + H_B \, \otimes \, I_Q + \sum_{j=1}^{N} H_I^{(j)} = \\ & I_B \, \otimes \sum_{j=1}^{N} \Omega^{(j)} \, \sigma^z_{(j)} + \omega_0 \, a^\dagger a \, \otimes \, I_Q + \frac{g}{\sqrt{N}} \sum_{j=1}^{N} \Big(a \, \otimes \, \sigma^+_{(j)} \, + a^\dagger \, \otimes \, \sigma^-_{(j)}\Big). \end{aligned} \tag{A.20}$$

One point which is important to stress is that the terms which we ignore in Eq.(A.19) and Eq.(A.20) are the so called counter-rotating terms. This approximation is known as the rotating-wave-approximation. In the rotating-wave-approximation we ignore energy non-conserving terms in which the emission (absorption) of a quantum of a quantized field is accompanied by the transition of one qubit from its lower (upper) to its upper (lower) state.

So far we have discussed N non-identical qubits interacting with one-mode of the quantized Bose field. Our aim is now to discuss the interaction of a system of N identical qubits with energy gap ($\Omega = \omega_2 - \omega_1$), with an infinite number of harmonic oscillators which defines the reservoir. Let $b_k^\dagger$ and b_k be the creation and annihilation operators of the $k-th$ harmonic oscillator of frequency ω_k. The total Hamiltonian, i.e., the Hamiltonian of the combined system of the reservoir and the N identical qubits interacting with the reservoir reads

$$I_B \, \otimes \Omega \sum_{j=1}^{N} \sigma^z_{(j)} + \sum_k \omega_k \, b_k^\dagger \, b_k \, \otimes \, I_Q + \frac{g}{\sqrt{N}} \sum_{j=1}^{N} \sum_k \Big(b_k \, \otimes \, \sigma^+_{(j)} + b_k^\dagger \, \otimes \, \sigma^-_{(j)}\Big). \tag{A.21}$$

In the Eq.(A.21) the first term is the free Hamiltonian of N identical qubits, the second term is the free reservoir Hamiltonian and the third term is the interaction Hamiltonian between the reservoir and the N identical qubits. Notice that we shift the zero of energy for each qubits, as we did before, and we are assuming the rotating-wave-approximation, where $\frac{g}{\sqrt{N}}$ is the j-th qubit, k-th harmonic oscillator coupling constant.

We can also use a different interaction Hamiltonian, as the one introduced by Di Vicenzo [62]. This author proposed a soluble model to study the influence of decoherence in quantum

computers, with the following model describing a system of one qubit coupled to a reservoir of harmonic oscillators:

$$
\begin{aligned}
&I_B \otimes H_Q + H_B \otimes I_Q + H_I = \\
&I_B \otimes \Omega\,\sigma^z + \sum_k \omega_k\, b_k^\dagger\, b_k \otimes I_Q + g \sum_k \left(b_k^\dagger + b_k\right) \otimes \sigma^z,
\end{aligned}
\tag{A.22}
$$

where Ω is the usual energy level spacing of the qubit, $b_k^\dagger$ and b_k are respectively the bosonic creation and annihilation operators of the harmonic oscillators. Notice the particular coupling between the reservoir and the qubit, that allows the loss of quantum coherence induced by the reservoir without affecting the qubit. There are two straightforward generalizations for this model. The first one is the introduction of a mode-dependent coupling constant [63]. Therefore we have

$$
\begin{aligned}
&I_B \otimes H_Q + H_B \otimes I_Q + H_I = \\
&I_B \otimes \Omega\,\sigma^z + \sum_k \omega_k\, b_k^\dagger\, b_k \otimes I_Q + \sum_k \left(\lambda_k\, b_k^\dagger + \lambda_k^*\, b_k\right) \otimes \sigma^z.
\end{aligned}
\tag{A.23}
$$

Other straightforward generalization is to introduce N identical qubits and the Hamiltonian of the composed system reads

$$
\begin{aligned}
&I_B \otimes H_Q + H_B \otimes I_Q + H_I = \\
&I_B \otimes \Omega \sum_{j=1}^{N} \sigma_{(j)}^z + \sum_k \omega_k\, b_k^\dagger\, b_k \otimes I_Q + \frac{g}{\sqrt{N}} \sum_{j=1}^{N} \sum_k \left(b_k^\dagger + b_k\right) \otimes \sigma_{(j)}^z.
\end{aligned}
\tag{A.24}
$$

Another generalization of the Hamiltonian given by Eq.(A.21) is not assume the rotating-wave-approximation in the interaction Hamiltonian. Without the rotating-wave-approximation, the interaction Hamiltonian between the N qubits and the reservoir of harmonic oscillators reads

$$
H_I = \frac{g}{\sqrt{N}} \sum_{j=1}^{N} \sum_k \left(b_k + b_k^\dagger\right) \otimes \left(\sigma_{(j)}^+ + \sigma_{(j)}^-\right). \tag{A.25}
$$

To conclude this appendix we should point out that through the paper we used the simple model where the interaction Hamiltonian between the two-level system and the scalar field is linear in both field and qubit and is given by

$$
H_I = \lambda \left(m_{21}\,\sigma^+ + m_{12}\,\sigma^- + \sigma^z (m_{22} - m_{11})\right) \otimes \varphi(x), \tag{A.26}
$$

where $m_{ij} = \langle\, i \,|\, m(0) \,|\, j \,\rangle$, and λ is a small coupling constant. The Bose field $\varphi(x)$ can be expanded as

$$
\varphi(x) = \sum_{\mathbf{k}} \left(a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(t, \mathbf{x})\right). \tag{A.27}
$$

The modes $u_{\mathbf{k}}(t, \mathbf{x})$ form a basis in the space of solutions of the Klein-Gordon equation. It's convenient to restrict the $u_{\mathbf{k}}(t, \mathbf{x})$ to the interior of a three dimensional torus of side L (i.e., choose periodic boundary conditions). Then

$$u_{\mathbf{k}}(t, \mathbf{x}) = (2L^3\omega)^{-1/2} e^{i\mathbf{k}.\mathbf{x} - i\omega t} \tag{A.28}$$

where

$$k_i = 2\pi j_i / L \quad j_i = 0, \pm 1, \pm 2 ... \;\; i = 1, 2, 3. \tag{A.29}$$

It is possible to show that the interaction Hamiltonian defined by Eq.(A.26) is equivalent to the interaction hamiltonian given by $H_I = m(\tau)\varphi(x(\tau))$. This model is known as the Unruh-Dewitt detector. The detector is an idealized point-like object with internal degrees of freedom defining two energy levels. Different coupling between the field and the two-level system was analyzed by Hinton [64].

References

[1] G. Gamow, Z. Phys. **51**, 204 (1928).

[2] V. Weisskopf and E. Wigner, Z. Phys. **63**, 54 (1930).

[3] L. Fonda, G. C. Ghirardi and A. Rimini, Rep. Prog. Phys. **41**, 587 (1978).

[4] A. Peres, Ann. Phys. **129**, 33 (1980).

[5] M. Namiki, S. Pascazio and H. Nakazato, *"Decoherence and Quantum Measurements"*, World Scientific, Singapure (1997).

[6] B. Misra and E. C. G. Sudarshan, Jour. Math. Phys. **18**, 756 (1977).

[7] C. B. Chiu, E. C. G. Sudarshan and B. Misra, Phys. Rev. **D16**, 520 (1977).

[8] J. von Neumann,*"Mathematical Foundations of Quantum Mechanics"*, Princeton University Press, New York (1983).

[9] Plato, *"The Collected Dialogues"*, E. Hamilton and H. Cairns (Editors), Princeton University Press, N.Y (1989), pp. 921-956.

[10] L. A. Khalfin, Sov. Phys. JETP **6**, 1053 (1958).

[11] D. Home and M. A. B. Whitaker, J. Phys. **A25**, 657 (1992).

[12] D. Home and M. A. B. Whitaker, Phys. Lett. **A173**, 327 (1993).

[13] H. Nakazato, M. Namiki, S. Pascazio and H. Rauch, Phys. Lett. **A199**, 27 (1995).

[14] A. O. Caldeira and A. J. Legget, Ann. Phys. **149**, 374 (1983).

[15] E. Joos, Phys. Rev. **D29**, 1626 (1984).

[16] K. H. Li, Phys. Rep. **134**, 1 (1986).

[17] H. Grabert, P. Schramm and G. L. Ingold, Phys. Rep. **168**, 115 (1988).

[18] D. Merchede, Phys. Rep. **211**, 201 (1992).

[19] R. R. Puri, *"Mathematical Methods of Quantum Optics"*, Oxford University Press (2001).

[20] H. P. Breuer and F. Petruccione, *"The Theory of Open Quantum Systems"*, Oxford Univesity Press (2003).

[21] A. Einstein, Phys. Z **18**, 121 (1917).

[22] B. F. Svaiter and N. F. Svaiter, Phys. Rev. **D46**, 5267 (1992).

[23] B. F. Svaiter and N. F. Svaiter, Phys. Rev. **D47**, 4802 (1993) (erratum).

[24] P. L. Knight and L. Allen, *Concepts in Quantum Optics*, Pergamon Press Inc. N.Y. (1983).

[25] H. Dekker, Phys. Rep. **80**, 1 (1981).

[26] L. H. Ford, N. F. Svaiter and M. L. Lyra, Phys. Rev. **A49**, 1378 (1994).

[27] D. Meschede, Phys. Rep. **211**, 201 (1992).

[28] V. A. De Lorenci, R. De Paola and N. F. Svaiter, Class. Quant. Grav. **20**, 4241 (2000).

[29] R. De Paola and N. F. Svaiter, Class. Quant. Grav. **18**, 1799 (2001).

[30] W. G. Unruh, Phys. Rev. **D14**, 870 (1976).

[31] B. S. DeWitt, Phys. Rep. **196**, 295 (1976).

[32] N. D. Birrel and P. C. W. Davies, *"Quantum Fields in Curved Space"*, Cambridge University Press, Cambridge (1984), pp. 48-59.

[33] V. A. De Lorenci and N. F. Svaiter, Int. Jour. Mod. Phys. **A14**, 717 (1999).

[34] V. A. De Lorenci and N. F. Svaiter, Found. Phys. **29**, 1233 (1999).

[35] A. Venugopalanand and R. Ghosh, Phys. Lett. **A204**, 11 (1995).

[36] A. K. Peti, Phys. Lett. **A215**, 7 (1996).

[37] L. D. Landau and R. Peierls, Z. Phys. **69**, 56 (1931).

[38] L. D. Landau and E. Lifshitz,*"Mecânica Quântica - Teoria Não-Relativística"*, Editora Mir, Moscou (1974), pp. 188-191.

[39] V. Fock and N. Krylov, J. Phys. (URSS) **11**, 112 (1947).

[40] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).

[41] D. Bohm,*"Quantum Theory"*, Dover Publications, Inc, New York (1979).

[42] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).

[43] *"Handbook of Mathematical Functions"*, M. Abramowitz and I. Stegun (Editors), Dover Publications, New York (1965).

[44] B. F. Svaiter and N. F. Svaiter, Class. Quant. Grav. **11** 347 (1994).

[45] J. Dalibard, J. Dupont Roc and C. Cohen Tannoudji, J. Phys. **43**, 1617 (1982).

[46] J. Dalibard, J. Dupont Roc and C. Cohen Tannoudji, J. Phys. **45**, 637 (1984).

[47] F. Oberhettinger, *"Table of Fourier Transform and Fourier Transform of Distributions"*, Springer-Verlag, Berlin (1980).

[48] N. F. Svaiter, Physica **A345**, 517 (2004).

[49] N. F. Svaiter, Physica **A368**, 111 (2006).

[50] D. F. Mundarain and J. Stephany, Phy. Rev. **A73**, 042113 (2006).

[51] D. Z. Albert, Phys. Lett. **A98**, 249 (1983).

[52] R. P. Feynman, Int. Jour. Mod. Phys. **21**, 467 (1982).

[53] R. P. Feynman, Found. Phys. **16**, 507 (1986).

[54] D. Deutsch, Proc. R. Soc. Lond. **A400**, 97 (1985).

[55] A. Steane, Rep. Prog. Phys. **61**, 117 (1998).

[56] Z. Ficek and Z. Tanas, Phys. Rep. **372**, 369 (2002).

[57] R. H. Lehmberg, Phys. Rev. **A2**, 883 (1970).

[58] R. H. Lehmberg, Phys. Rev. **A2**, 889 (1970)

[59] P.W. Milonni and P. L. Knight, Phys. Rev. **A10**, 1096 (1974).

[60] P.W. Milonni and P. L. Knight, Phys. Rev. **A11**, 1090 (1975).

[61] E. T. Jaynes and F. W. Cummings, Proc. I. E. E. **51**, 89 (1963).

[62] D. P. DiVicenzo, Phys. Rev. **A51**, 1015 (1995).

[63] F. Benatti and R. Floreani, *"Open Quantum Dynamics: Complete Positivity and Entanglement"*, quant. ph/0507271.

[64] K. J. Hinton, Class. Quant. Grav. **1**, 27 (1984).